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ON THE STABILITY LOSS OF WEAKLY NON-UNIFORM FLOWS IN EXTENDED REGIONS. THE FORMATION OF TRANSVERSE OSCILLATIONS OF A TUBE CONVEYING A FLUID†

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The stability of weakly non-uniform flows is investigated by the WKB method. Reflecting and refracting wave boundary conditions on the Stokes lines are considered. Approaches to analysing the situation when the turning points do not lie on the real axis and the eigenfunctions are fairly complex are discussed.

As an example, the instability of transverse oscillations of a weakly inhomogeneous tube conveying a liquid is considered. It is established that under certain conditions a growing oscillatory mode can emerge; it represents a chain of four waves if the convective instability condition is satisfied, while the local condition of absolute instability is not satisfied at any point. A qualitative analysis of the conditions is carried out in terms of frequencies and wave numbers.

A "GLOBAL STABILITY CRITERION" has been presented in a number of papers [1, 2] and used to investigate specific flows. This criterion is based on finding an eigenfunction associated with the presence of a multiple turning point in the complex plane (and in more detailed considerations, with two close turning points). For such an instability there is a section on the real axis where a local absolute [3] instability condition is satisfied. The same criterion was obtained by another method in [4] by considering the development of non-stationary perturbations as a case of one of the possible scenarios of the transition to instability.

Note that the term "local" is more appropriate for an instability associated with the growth of a perturbation representing two waves blocked by two close turning points. The term "global instability" was used previously [3, 5] when considering the stability of uniform flows with boundary conditions at the ends of a long interval. This term denoted the existence of a growing perturbation, consisting of a set of waves, two of which pass through the entire interval and are reflected at both its ends (as distinct from boundary or one-sided instability, when the perturbation evolves around one end of the interval).

Many papers on the stability of weakly non-uniform states and flows (for example, [6–8]), have explained that perturbation modes responsible for the transition to instability are chains of waves with different wave numbers that transform into one another.

A similar result was obtained [9] in an analysis of general criteria for eigenfunction formation in weakly non-uniform flows when points are present on the real axis at which arbitrary boundary conditions are inserted. It was assumed [9] that the wave reflection and refraction coefficients specified by the boundary conditions are finite, i.e. have a finite limit as the length-scale tends to infinity. If the boundary conditions are associated with the Stokes

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phenomenon [7, 10], i.e. with discontinuous variation of the coefficients in the WKB representation of the solution, then the possibility of applying the result obtained in [9] needs some clarifications if the corresponding turning point does not lie on the real x axis. These clarifications are given in Sec. 1.

In Sec. 2 the problem of the transverse oscillations of a weakly non-uniform tube conveying a fluid is considered. Unlike the situations considered in [1, 2], the oscillation mode corresponding to growing perturbations is a chain of four waves, and the local condition of absolute instability is not satisfied anywhere. It is also interesting to note that the formation of the eigenfunction in this example is not associated with the joining of anti-Stokes lines ("level lines") of different turning points, as is sometimes the case for second-order equations and in some other situations [6, 7, 11].

1. In stability investigations one can in many cases assume that the time-dependence of the perturbation has the form $(i\omega t)$, after which the study of small perturbation behaviour reduces in typical situations to the study of boundary-value problems for a system of ordinary linear differential equations. If the coefficients of a linear uniform system of ordinary differential equations depend on x/L, where the length-scale L of the range of variation of x is assumed to be large, then over finite ranges of variation of x the general solution, according to the WKB method [7, 10], can be represented as a linear combination of elementary WKB solutions with coefficients C_m

$$\sum_m C_m A_m\left(\frac{x}{L}\right)\exp\left[i\int_{x_0}^{x} k_m\left(\omega,\frac{\xi}{L}\right)d\xi\right] \tag{1.1}$$

Here $k_m(\omega,\ x/L)$ is a branch of the multivalued function $k(\omega,\ x/L)$ defined from the dispersion relation $\Phi(\omega, k,\ x/L)=0$ of the system under consideration, in which the variable x/L occurs as a parameter.

As in [9], we shall assume that there is (or that it is sufficient to consider) just a finite number of branches $k_m(\omega,\ x/L)$ $(m=1, 2, \dots, N)$, as is the case for solutions of differential equations. Moreover, we shall assume that for all ω of interest, there are only isolated branch points for the function $k(\omega, x/L)$ in the complex x plane. For every specified m the vector $A_m(x/L)$ (with the number of components equal to the number of unknowns) is found by the WKB method in the form of an asymptotic series [10].

For solutions of the differential equations, there is not in general a single asymptotic representation of the general solution of the form (1.1) for the entire real x axis or for the entire complex x plane. One can, for example, recall that an asymptotic representation (1.1) can be obtained for every section of the real x axis in which the imaginary parts of all the roots of $k_j(\omega,\ x/L)$ are different [10]. In the complex x plane the discontinuities in the asymptotic representations are specified on certain lines called the Stokes lines.

The Stokes lines emerge from the turning points, which are defined as the branch points of the function $k(\omega,\ x/L)$ in the complex x plane for given ω. Below, for simplicity, we shall consider simple turning points, at which for the given ω the values of two wave numbers are the same: $k_p=k_q$. We shall assume as usual [7, 10] that in a neighbourhood of such a point there is significant interaction only between those elementary WKB solutions corresponding to k_p and k_q (their interaction being resonant at the turning point), and that it is independent of the other elementary WKB solutions. This means that in the neighbourhood of the turning point the branching WKB solutions behave in the same way as solutions of an equation (or system of equations) of the second order with a dependence $k(\omega,\ x)$ approximating in this neighbourhood the branches $k_p(\omega,\ x/L)$ and $k_q(\omega,\ x/L)$. Here from the branch point $x=A$ three Stokes lines emerge inside each of the sectors, in which

$$\operatorname{Im}\int_A^x [k_p(\omega,\xi/L)-k_q(\omega,\xi/L)]d\xi \neq 0$$

On each Stokes line a discontinuity occurs in the coefficient of the term in (1.1) containing the smaller (in absolute value) exponent among those corresponding to k_p and k_q, where we have put $x_0 = A$, and the magnitude of the discontinuity is proportional to the coefficient of the term with the larger exponent.

Thus, in the neighbourhood of the point A, in different sectors there are different asymptotic representations of the solution for the transition from one sector to another. C_p and C_q are related to one another by linear relations with finite coefficients (the Stokes coefficients) [7, 10], if as a lower limit of integration in the exponents in (1.1) we take the turning point $x_0 = A$.

If there are turning points not lying on the real x axis, one can try to deform the x axis so that the new curvilinear x' axis passes through the turning points which have to be taken into account when constructing the solution. The above-mentioned relations with finite coefficients are specified at each of these turning points in the form of boundary conditions linking solutions on both sides of the x' line with respect to the turning point. This is the normal procedure in the case of second-order equations and in many other cases [6, 7, 11].

In order to use this method to construct a solution, one must be able to deform the x axis in such a way as to obtain a line that does not intersect any Stokes lines emerging from any turning points except at the turning points through which it passes. If such a deformation of the x axis is possible, one can use the procedure presented in [9] to construct the eigenfunctions, and we do this below in Sec. 2.

2. We consider, as an example of instability with growing perturbations consisting of more than two elementary WKB solutions ("waves"), the instability of a tube with fluid flowing through it, appearing in the form of transverse oscillations. The transverse displacement of the tube $w(x, t)$ along the direction of one of the coordinate axes orthogonal to the position of the unperturbed tube axis, taken to be the x axis, satisfies the equation [12]

$$\rho_1 \frac{\partial^2 w}{\partial t^2} + \rho_2 \left(\frac{\partial}{\partial t} + \upsilon \frac{\partial}{\partial x} \right)^2 w = -Fw - D\frac{\partial^4 w}{\partial x^4} \tag{2.1}$$

Here ρ_1 and ρ_2 are the mass of the tube and fluid per unit length, $-Fw$ is the elastic restoring force, and D is the bending stiffness of the tube. The oscillations described by these equations were studied [12–14] for the case of a homogeneous tube when all the coefficients in (2.1) are constants and $F = 0$. It was shown [14] that when $\rho_1/\rho_2 > 1/8$ the instability is absolute, while for $\rho_1/\rho_2 < 1/8$ the instability is convective. It was also shown that when the convective instability criterion is satisfied the global instability criterion in the sense of [5] is satisfied, i.e. for boundary conditions of general form, imposed at the ends of a fairly long tube, a growing mode of fundamental oscillations occurs, associated with the passage of bending waves along the entire tube and their reflection from its ends.

Here we consider the case of an inhomogeneous tube with slowly varying parameters and we show that under certain conditions a mode of growing oscillations, in the form of a chain of four waves, can emerge when the local convective instability condition is satisfied at every point.

We write the dispersion relation corresponding to (2.1) in the form of a solution for ω

$$\omega = \Omega_{1,2}(k, x) \equiv Uk \mp \sqrt{Qk^4 - Pk^2 + R} \tag{2.2}$$

$$U = \frac{\rho_2 \upsilon}{\rho_1 + \rho_2}, \quad P = \frac{\rho_1 \rho_2 \upsilon^2}{(\rho_1 + \rho_2)^2}, \quad Q = \frac{D}{\rho_1 + \rho_2}, \quad R = \frac{F}{\rho_1 + \rho_2}$$

We first assume that $Q = A + Bx^2$, that U, P, R, A and B are positive constants, and that apart from U they are all sufficiently small.

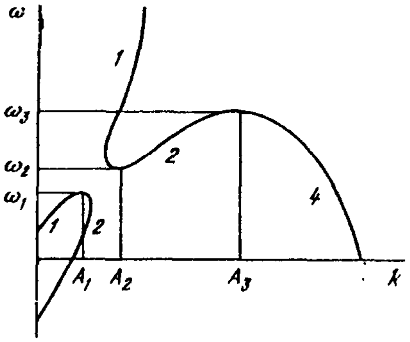

FIG. 1.

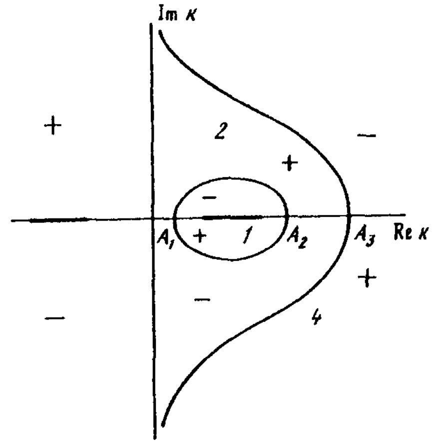

FIG. 2.

The dependence of ω on real k and x is odd. Suppose that at $x=0$ it has the form shown in Fig. 1. In this case, all six branch points of the function $k(\omega)$ correspond to real values of $\omega=\pm\omega_i$ $(i=1, 2, 3)$. This means that the plot shown in Fig. 1 ensures the satisfaction of the local convective instability criterion. As x increases the plot of $\omega(k)$ changes, with ω_1 increasing and ω_2 and ω_3 decreasing. For some values of x the branches of the plot merge, and then ω_3 decreases to negative values. Here, of course, the local absolute instability condition is not satisfied for any x.

The dependence of Im $\Omega_1(k, 0)$ on complete k is characterized by Fig. 2 which shows the lines Im $\Omega_1(k, 0)=0$. The plus and minus signs indicate the domains of positive and negative Im Ω_1. The numbers 1–4 in Figs 1 and 2 denote the branches of the function $k(\omega)$ in such a way that as Im $\omega\to\infty$ the quantity $k_m(\omega)$ tends to infinity whilst staying in the mth quadrant of the complex k plane. Thus k_1 and k_2 correspond to waves propagating to the right, and k_3 and k_4 to waves propagating to the left. The bold lines along the real axis in Fig. 2 depict cuts along which Im Ω_1 is discontinuous. For the second branch $\omega=\Omega_2(k, 0)$ we have the equality Im $\Omega_2(k, 0)=$ Im $\Omega_1(-k, 0)$.

One can verify that from the branches of the function $k(\omega)$ only the $k_1(\omega)$ branch with Im $\omega>0$ has values with imaginary parts of different signs. This means that only this wave with Im $\omega>0$ can experience exponential growth in the direction of its own propagation in accordance with the presence in the solution of the factor $\exp[i\int k_1(\omega, \xi/L)d\xi]$. For all ω with Im $\omega>0$ all the remaining waves experience spatial damping in their direction of propagation.

Hence the chain of mutually transforming waves, necessary [9] to form the eigenfunction when Im $\omega>0$, can exist only for ω corresponding to values of k from domain 1 in Fig. 2. One can verify that the above domain of variation of ω contracts into a point when the difference $\omega_2-\omega_1$ tends to zero, this difference depending on the values of constants occurring in the statement of the problem. If the quantity $\omega_2-\omega_1$ is small, the segment of the real x axis along which amplification of the k_1 wave occurs is small, and the quantity Im $k_1(\omega, x)$ is also small on this segment.

It follows from the above that if $\omega_2-\omega_1$ is small, one can expect the formation of growing eigenfunctions with very small Im ω. Hence, we first consider the propagation of waves corresponding to real ω in the interval $\omega_1<\omega<\omega_2$, because only for these values of ω can a k_1 wave experience spatial growth. In this case, the turning points lie on the real x axis. For small x these will be points corresponding to either A_1 or A_2 in Figs 1 and 2 at which $k_1=k_2$, while for finite x (i.e. not connected to the smallness of the difference $\omega_2-\omega_1$) these will be turning points corresponding to the point A_3 in Fig. 2 at which $k_2=k_4$. Figure 3 shows the x axis with two pairs of symmetrically situated turning points A_3', A_3 and A_2', A_2. The wave numbers of waves propagating along the corresponding intervals of the x axis are shown above and below,

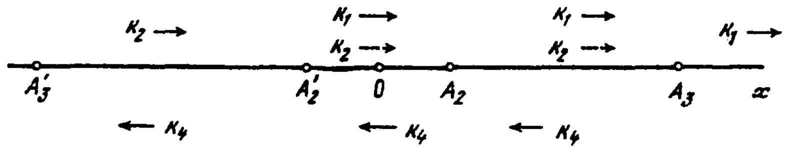

FIG. 3.

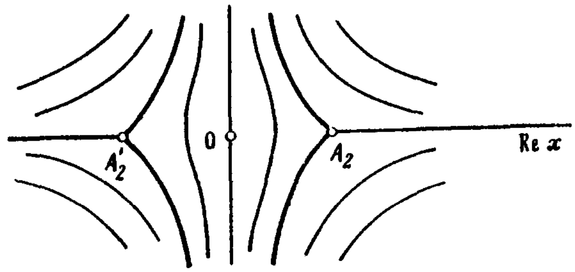

FIG. 4.

and the arrows indicate the direction of propagation. Outside the section $A_3'A_3$ the waves corresponding to k_2 and k_4 experience exponential damping, and they are not shown in Fig. 3. Consider the interaction of waves propagating to the right, corresponding to k_1 and k_2 in a small domain in the complex x plane that includes the points A_2' and A_2. The Im $\int[k_2(\omega,\ \xi/L)-k_1(\omega,\ \xi/L)]d\xi$ contour lines are shown in Fig. 4 with Im $\omega=0$. At the point A_2' the k_2 wave for $x<A_2'$ changes, with finite non-zero coefficients, into waves corresponding to complex k_1 and k_2 in the $A_2'<x<A_2$ domain, the former being amplified as x increases, and the latter weakened. At the point A_2 these waves are transformed into waves with real k_1 and k_2, which propagate to the right of A_2. If in the left half of the neighbourhood of A_2 one neglects the damped k_2 wave, then to the right of A_2 the amplitudes of the two waves can be assumed to be equal.

Thus, as a result of passing through the section $A_2'A_2$, the k_2 wave experiences spatial amplification, with the dominant part of the amplitude variation of the wave being given by the quantity

$$\exp\left[-\mathrm{Im}\int_{A_2'}^{A_2} k_1(\omega,\xi/L)d\xi\right]$$

In this process a k_1 wave is generated, which, without interacting with other waves, then leaves the section $[A_3'A_3]$ and will be ignored. The k_2, k_4 wave system reflected from the points A_3 and A_3' is amplified in the section $A_2'A_2$ for $\omega_1<\omega<\omega_2$. The length of the section $A_2'A_2$ is, by assumption, much smaller than the length of $A_3'A_3$, and the spatial wave amplification index corresponding to k_1 is also small, while for small Im $\omega>0$ the remaining waves experience spatial damping proportional to Im ω Hence it is easy to select a small Im $\omega>0$ so that the total net amplification and damping in the k_2, k_1, k_2, k_4 wave chain cancels out, thus, making the construction of an eigenfunction possible.

It is obvious that the existence of an eigenfunction in the case under consideration is associated with the presence of all four turning points. If, when the points A_3' and A_3 exist, one varies the values of the constant coefficients in the dispersion equation, then instability arises when the local convective instability condition is satisfied in the neighbourhood of $x=0$ together with the appearance of the points A_2' and A_2 on the real x axis. Here the growing fundamental oscillations occur immediately in the finite section $A_3'A_3$ and consist, as has previously been stated, of four waves.

For other dependences of Q on x the turning points A_3' and A_3 can be complex for real ω and do not give rise to effective reflection conditions for k_2 and k_4 waves (for example, when $Q = A + B\exp(-x^2/L^2)$, the points A_3' and A_2 can lie on the imaginary x axis). In this case, instability does not occur when the section $A_2'A_2$ appears on the real x axis with the local convective instability condition.

We note in conclusion that effects associated with the non-analyticity coefficients of the equations have not been considered in this paper.

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